ON USING DURBIN'S h-TEST TO VALIDATE
THE PARTIAL-ADJUSTMENT MODEL

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No.21 - November 1985

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ISSN 0157-0188

ISBN 0 85834 607 9

1. Introduction

The partial adjustment (PA) model, originally proposed by Nerlove (1958) has had long and continued usage in empirical research. Its attraction lies in its simplicity - it involves few parameters and can be estimated by ordinary least squares (OLS). It still has an important role in situations where inadequate data make it necessary to invest in a simple model. However, if simple models are to be used to represent complex reality, it is necessary to have valid diagnostic procedures available to detect when the models are inappropriate.

The PA model may be formulated as follows:

$$Y_t^* = \alpha + \beta X_{t-1} \tag{1.1}$$

$$Y_t - Y_{t-1} = (1-\gamma)(Y_t^* - Y_{t-1}) + v_t, \tag{1.2}$$

where Y_t^* is the desired or optimal level of the variable Y_t (often production or investment), v_t is i.i.d $N(0,\sigma^2)$ and $0 \leqq \gamma < 1$. When Y_t^* is eliminated from (1.1) and (1.2), the estimating equation

$$Y_t = \alpha(1-\gamma) + \beta(1-\gamma)X_{t-1} + \gamma Y_{t-1} + v_t \tag{1.3}$$

is obtained.

A common method of diagnostically validating the PA model is to regress Y_t on X_{t-1} and Y_{t-1} and examine the residuals for autocorrelation using Durbin's (1970) h-statistic. There is theoretical justification for this procedure.

Suppose the PA model is nested in the more general Partial Adjustment-Adaptive Expectations (PAAE) model, in which X_{t-1} is replaced by X_t^*, given by

$$X_t^* - X_{t-1}^* = (1-\delta)(X_{t-1}-X_{t-1}^*).^{(1)} \qquad (1.4)$$

The estimating model then becomes

$$Y_t = b_0 + b_1X_{t-1} + b_2Y_{t-1} + b_3Y_{t-2} + u_t \ , \qquad (1.5)$$

where

$$b_0 = \alpha(1-\gamma)(1-\delta),\ b_1 = \beta(1-\gamma)(1-\delta),\ b_2 = \gamma+\delta,\ b_3 = -\gamma\delta$$

and

$$u_t = v_t - \delta v_{t-1}, \quad (0 \leqq \delta < 1). \qquad (1.6)$$

The PA model may be validated by testing the hypothesis $\delta = 0$ in (1.5) and (1.6). These equations represent a dynamic model with a first-order moving average disturbance and Breusch (1978) has shown that the h-test is asymptotically equivalent to a Lagrange Multiplier (LM) test of the hypothesis $\delta = 0$.

However, an important feature of the PAAE model is that there is a non-linear restriction on the parameters.

From the definition of b_2 and b_3 above,

$$b_3 = -\delta(b_2-\delta). \qquad (1.7)$$

Use of Durbin's h-test ignores this restriction and therefore results in a loss of power.

The purpose of this note is to derive a LM test which does incorporate the restriction (1.7) and to demonstrate its relationship to the h-test.

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(1) Equation (1.4) implies $X_t^* = (1-\delta)\sum_{i=0}^{\infty}\delta^i X_{t-i-1}$.

2. A Lagrange Multiplier Test

When X_{t-1} is replaced by $X_t^* = (1-\delta)\sum_{i=0}^{\infty} \delta^i X_{t-i-1}$ in (1.3), the PAAE model becomes

$$Y_t = \beta_0 + \beta_1\left\{\sum_{i=0}^{\infty} \delta^i X_{t-i-1}\right\} + \beta_2 Y_{t-1} + v_t, \tag{2.1}$$

where

$$\beta_0 = \alpha(1-\gamma), \quad \beta_1 = \beta(1-\gamma)(1-\delta), \quad \beta_2 = \gamma$$

and the symbols have been chosen so that under the null $\delta = 0$, $\beta_i = b_i$ $(i=0,1,2)$. The restriction (1.7) is implicit in this formulation.

If we define $\theta = (\beta_0, \beta_1, \beta_2)'$, then the likelihood function $L(\delta, \theta, \sigma^2)$ is given by

$$L = -\frac{N}{2}\ln(2\pi) - \frac{N}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}v'v, \tag{2.2}$$

where N is the sample size and $v = (v_1, v_2, \ldots, v_N)'$. The Lagrange Multiplier statistic LM has the form

$$LM = N^{-1}\left(\frac{\hat{\partial L}}{\partial\delta}\right)'\left(\hat{I}_{11} - \hat{I}_{12}\hat{I}_{22}^{-1}\hat{I}_{21}\right)^{-1}\left(\frac{\hat{\partial L}}{\partial\delta}\right),$$

where (see, for example, Breusch and Pagan (1980)) $I_{ij} = -\text{plim}(N^{-1}\partial^2 L/\partial\phi_i\partial\phi_j')$ with $\phi_1 = \delta$, $\phi_2 = \theta$, and '^' indicates evaluation under the null hypothesis. When the null is true, LM is asymptotically distributed as χ_1^2. Doran (1985) has shown that when testing the PA model, LM takes the form

$$LM = \hat{\sigma}^2 N^{-1}\left(\frac{\hat{\partial L}}{\partial\delta}\right)^2 / \text{plim}(N^{-1}\text{RSS}), \tag{2.3}$$

where RSS is the residual sum of squares which is obtained when $\frac{\partial\hat{v}}{\partial\delta}$ is regressed on $\frac{\partial\hat{v}}{\partial\theta}$.

From (2.1)

$$\frac{\partial \hat{v}}{\partial \delta} = -\hat{\beta}_1 X_{-2} ,$$

$$\frac{\partial \hat{L}}{\partial \delta} = -\frac{1}{\hat{\sigma}^2} \hat{v}' \frac{\partial \hat{v}}{\partial \delta} = \frac{\hat{\beta}_1}{\hat{\sigma}^2} \hat{v}' X_{-2} ,$$

$$\frac{\partial \hat{v}}{\partial \theta} = -[j,\ X_{-1},\ Y_{-1}],\ j \text{ being a vector of ones.}$$

Thus, in finite samples

$$LM = (N\hat{\sigma}^2)^{-1}\ \hat{\beta}_1^2 (\hat{v}'X_{-2})^2 / (N^{-1}\hat{\beta}_1^2\ RSS^*),$$

where $RSS^* = \hat{v}^{*\prime}\hat{v}^*$, $\hat{v}^*$ being the residual vector obtained when X_{-2} is regressed on j, X_{-1} and Y_{-1}.

That is, on writing $N\hat{\sigma}^2 = \hat{v}'\hat{v}$,

$$LM = N(\hat{v}'X_{-2})^2 / [(\hat{v}'\hat{v})(\hat{v}^{*\prime}\hat{v}^*)].$$

Also, because $\hat{v}^*$ can be written in the form $\hat{v}^* = X_{-2} - (j,\ X_{-1},\ Y_{-1})\ \hat{c}$ for some vector $\hat{c}$,

$$\hat{v}'\hat{v}^* = \hat{v}'X_{-2} - \hat{v}'(j,\ X_{-1},\ Y_{-1})\hat{c} = \hat{v}'X_{-2}$$

by the OLS properties of $\hat{v}$, which ensure that $\hat{v}'j = \hat{v}'X_{-1} = \hat{v}'Y_{-1} = 0$.

Finally then,

$$LM = Nr^2, \tag{2.4}$$

where r^2 is the 'R^2' which is obtained when $\hat{v}$ is regressed on $\hat{v}^*$.

The LM test of the PA model thus consists of the following simple procedure:

(i) regress Y on X_{-1} and Y_{-1} to obtain the residuals $\hat{v}$;

(ii) regress X_{-2} on X_{-1} and Y_{-1} to obtain the residuals $\hat{v}^*$;

(iii) regress $\hat{v}$ on $\hat{v}^*$ to obtain r^2;

(iv) compare Nr^2 with χ_1^2.

It should be noted that the additional information that $\delta \geqq 0$ can be utilised by constructing a one-sided test in which $N^{\frac{1}{2}}r$ is compared with the standardised normal random variable, with the null being rejected when $N^{\frac{1}{2}}r$ exceeds the positive critical value.

The above procedure can be justified intuitively by recognising that the three regressions are equivalent to estimating the coefficient β_3 in the multiple regression model

$$Y_t = \beta_0 + \beta_1 X_{t-1} + \beta_2 Y_{t-1} + \beta_3 X_{t-2} + v_t.$$

A 'large' r^2 is directly related to a 'large' $\hat{\beta}_3$. Now from (2.1), the PAAE model may be written as

$$Y_t = \beta_0 + \beta_1 X_{t-1} + \beta_2 Y_{t-1} + \beta_1 \delta X_{t-2} + \text{(higher orders in } \delta) + v_t$$

and the coefficient of X_{t-2} is seen to be proportional to δ.

3. Connection Between the LM and h Tests

In order to recognise the role that the non-linear restriction (1.7) plays in the LM statistic, it will be convenient to consider the PAAE model in the form (1.5) and (1.6). The presence of the restriction implies that $\partial u/\partial \delta$ is non-zero.

Expressing the moving-average process (1.6) in the form

$$u = M(\delta)v^{(2)} \tag{3.1}$$

where $M_{ii} = 1$, $M_{i,i-1} = -\delta$ and all other elements of M are zero, we have

$$\frac{\partial v}{\partial \delta} = \frac{\partial M^{-1}}{\partial \delta} u + M^{-1} \frac{\partial u}{\partial \delta} = - M^{-1} \frac{\partial M}{\partial \delta} M^{-1} u + M^{-1} \frac{\partial u}{\partial \delta} .$$

Thus

$$\frac{\partial \hat{v}}{\partial \delta} = V\hat{u} + \frac{\partial \hat{u}}{\partial \delta} \tag{3.2}$$

where $V_{i,i-1} = 1$ and otherwise $V_{ij} = 0$.

Substituting into (2.3),

$$LM = \hat{\sigma}^2 N^{-1} \left(\frac{1}{\hat{\sigma}^2} \hat{v}' \frac{\partial \hat{v}}{\partial \delta} \right)^2 / \text{plim}(N^{-1} RSS)$$

$$= [\hat{u}'(V\hat{u} + \frac{\partial \hat{u}}{\partial \delta})]^2 / [N\hat{\sigma}^2 \ \text{plim}(N^{-1} \ RSS)], \tag{3.3}$$

where RSS is obtained by regressing $V\hat{u} + \partial\hat{u}/\partial\delta$ on $Z = (j, X_{-1}, Y_{-1})$. This expression shows how $\partial\hat{u}/\partial\delta$ enters both the numerator and denominator of LM.

Suppose now that the restriction is ignored, and consequently, $\partial u/\partial\delta$ is taken to be zero. Then the LM statistic, call it LM*, is given by

$$LM^* = (\hat{u}'V\hat{u})^2 / [N\hat{\sigma}^2 \ \text{plim}(N^{-1} \ RSS^*)]$$

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(2) It is clear $M(\hat{\delta}) = I_N$ and hence $\hat{u} = \hat{v}$.

where RSS* is obtained by regressing $V\hat{u}$ on Z. That is,

$$RSS^* = \hat{u}'V'[I - Z(Z'Z)^{-1}Z']V\hat{u}.$$

From the definition of V,

$$\hat{u}'V\hat{u} = \sum_{t=2}^{N} \hat{u}_{t-1}\hat{u}_t = N\hat{\sigma}^2\hat{\rho},$$

$$\text{plim } N^{-1}(\hat{u}'V'V\hat{u}) = \text{plim } N^{-1}\sum_{t=1}^{N-1}\hat{u}_t^2 = \sigma^2,$$

$$\text{plim } N^{-1}(Z'V\hat{u}) = [0,\ 0,\ \sigma^2].$$

Thus,

$$\text{plim } N^{-1}\ RSS^* = \sigma^2\ \text{plim}[1 - N\hat{\sigma}^2(Z'Z)^{-1}_{3,3}] = \sigma^2[1 - NV(\hat{\beta}_2)].$$

For finite samples, we use

$$LM = \frac{N^2\hat{\sigma}^4\ \hat{\rho}^2}{N\hat{\sigma}^4\ [1 - N\hat{V}(\hat{\beta}_2)]} = \frac{N\hat{\rho}^2}{[1 - N\hat{V}(\hat{\beta}_2)]}\ . \qquad (3.4)$$

When the square-root of LM* is taken, enabling one-sided tests to be performed, we have precisely Durbin's h-statistic.

Preliminary Monte Carlo studies reported in Doran (1985) indicate that, at least in cases in which X_t has strong autocorrelation, the loss of power incurred from using the h-statistic instead of the LM statistic (2.4) is very considerable.

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