A counting problem is to determine for a given set, let’s call it $A$, how many elements it has. This is a very basic mathematical problem, and we have learned how to do this in kindergarten, for instance, when $A$ is the set of cookies in a jar, or the set of seats in a bus (see Figure 1).

On the other hand, mathematicians have developed a quite sophisticated and deep theory of counting, although some prefer to call it enumerative combinatorics (see Figure 2). Of course, the subject of this theory is not to count the cookies in one particular jar, but an abstract version of it, where we imagine a sequence of jars, together with a description of the cookies in the $n$-th jar, and the question is, how many cookies there are in the $n$-th jar. As an example, let $A_n$ be the set of sequences of zeros and ones of length $n$:

$A_1 = \{0, 1\}$,
$A_2 = \{00, 01, 10, 11\}$,
$A_3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$,
$A_4 = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$,

etc.

We see that $|A_1| = 2$, $|A_2| = 4$, $|A_3| = 8$, $|A_4| = 16$, and guess that there are $|A_n| = 2^n$ cookies in the $n$-th jar. A formal proof of this observation can be obtained by combining the following two basic counting principles:

**Sum rule:** If a set $A$ can be partitioned into two sets, say $B$ and $C$, that is, every element of $A$ is an element of either $B$ or $C$ (but not both), then $|A| = |B| + |C|$.

**Bijection rule:** If $A$ and $B$ are sets and there is a bijective function $f : A \to B$, that is, for every $b \in B$ there is exactly one element $a \in A$ with $f(a) = b$, then $|A| = |B|$.

The bijection can be illustrated by a method for counting the seats in the bus: if a group of 52 people enters the bus, everybody sits down and every seat is occupied by exactly one person, then the bus has exactly
52 seats. In order to apply this to our zero/one sequences, let \( B_n \) be the set of sequences of length \( n \) which start with 0, and let \( C_n \) be the set of sequences of length \( n \) which start with 1:

\[
B_1 = \{0\}, \quad C_1 = \{1\}
\]
\[
B_2 = \{00, 01\}, \quad C_2 = \{10, 11\},
\]
\[
B_3 = \{000, 001, 010, 011\}, \quad C_3 = \{100, 101, 110, 111\},
\]

etc.

We get bijections \( f : B_n \to A_{n-1} \) and \( g : C_n \to A_{n-1} \) by simply omitting the first element of the sequence, and by the bijection rule \( |B_n| = |C_n| = |A_{n-1}|.\) Then the sum rule implies \( |A_n| = |B_n| + |C_n| = 2|A_{n-1}|,\) and by induction, \( |A_n| = 2^n.\) The sum rule can be generalised to more than two sets:

**General sum rule:** If a set \( A \) can be partitioned into \( k \) sets, say \( B_1, B_2, \ldots, B_k,\) that is, every element of \( A \) is an element of exactly one of the sets \( B_i,\) then \( |A| = |B_1| + |B_2| + \cdots + |B_k|.\)

The above derivation of the formula \( |A_n| = 2^n \) is a simple example of a **recursion**, where a counting problem depending on a parameter \( n \) is reduced to the same problem for smaller values of the parameter. As another illustration, let \( A_n \) be the set of sequences of zeros and ones of length \( n,\) which do not contain two consecutive zeros:

\[
A_1 = \{0, 1\},
A_2 = \{01, 10, 11\},
A_3 = \{010, 011, 101, 110, 111\},
A_4 = \{0101, 0110, 0111, 1010, 1011, 1101, 1110, 1111\},
\]

etc.

We see \( |A_1| = 2, |A_2| = 3, |A_3| = 5, |A_4| = 8,\) and might find it a bit harder to guess a general formula for \( |A_n|.\) As before, we can partition \( A_n \) into the sets \( B_n \) and \( C_n,\) where \( B_n \) contains the sequences starting with 0 and \( C_n \) the sequences starting with 1:

\[
B_1 = \{0\}, \quad C_1 = \{1\}
\]
\[
B_2 = \{01\}, \quad C_2 = \{10, 11\},
\]
\[
B_3 = \{010, 011\}, \quad C_3 = \{101, 110, 111\},
\]
\[
B_4 = \{0101, 0110, 0111\}, \quad C_4 = \{1010, 1011, 1101, 1110, 1111\},
\]

etc.

Again, the sum rule gives \( |A_n| = |B_n| + |C_n|,\) and we can again try to set up bijections by omitting the first element of the sequence. If we start with an element of \( C_n,\) that is, a zero/one sequence of length \( n \) which starts with 1 and does not contain two consecutive zeros, then we obtain a sequence of length \( n - 1 \) which does not contain two consecutive zeros and any such sequence can be obtained this way. By the bijection rule, \( |C_n| = |A_{n-1}|.\) But if we start with an element of \( B_n,\) that is, a zero/one sequence of length \( n \) which starts with 0 and does not contain two consecutive zeros, then we obtain a sequence of length \( n - 1 \) which does not contain two consecutive zeros and starts with 1 (otherwise we have two consecutive zeros in the beginning of our sequence). By the bijection rule, \( |B_n| = |C_{n-1}| = |A_{n-2}|,\) and we obtain \( |A_n| = |A_{n-2}| + |A_{n-1}|.\) This does not quite give us a formula for \( |A_n|,\) but at least we can now determine \( |A_{10}|\) without writing down all the sequences of length 10: \( |A_5| = 13, |A_6| = 21, |A_7| = 34, |A_8| = 55, |A_9| = 89, |A_{10}| = 144.\) These numbers, which come up in many different contexts, are called **Fibonacci numbers** and they can be expressed by an explicit formula:

\[
|A_n| = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1}.
\]

**Subsets**

A rich source of counting problems are the subsets of a finite set, lets say the set \( \{1, 2, \ldots, n\} \) which we will denote by \( [n] \). The bijection principle tells us immediately, that the total number of subsets of \([n]\) is
2^n$, because we can set up a bijection between the zero/one sequences of length $n$ and the subsets of $[n]$, by mapping the sequence $a_1a_2\ldots a_n$ to the set $\{i \in [n] : a_i = 1\}$, for instance for $n = 3$:

- $000 \mapsto \emptyset$,
- $001 \mapsto \{3\}$,
- $010 \mapsto \{2\}$,
- $011 \mapsto \{2, 3\}$,
- $100 \mapsto \{1\}$,
- $101 \mapsto \{1, 3\}$,
- $110 \mapsto \{1, 2\}$,
- $111 \mapsto \{1, 2, 3\}$.

Let the \textit{binomial coefficient} $\binom{n}{k}$ be defined as the number of subsets of $[n]$ which have exactly $k$-elements (we will call these subsets $k$-subsets). For instance, $\binom{5}{3} = 10$, because the 3-subsets of $[5]$ are

$\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$, $\{2, 4, 5\}$, $\{3, 4, 5\}$.

Mapping every set to its complement provides a bijection between $k$-subsets and $(n-k)$-subsets, and by the bijection rule we have

$$\binom{n}{k} = \binom{n}{n-k}.$$  \hfill (1)

Let $B_{nk}$ be the set of all $k$-subsets of $[n]$, so that $|B_{nk}| = \binom{n}{k}$ and the sets $B_{n0}$, $B_{n1}$, $\ldots$, $B_{nn}$ form a partition of the set $A_n$ of all subsets of $[n]$. By the general sum rule, we obtain the identity

$$2^n = |A_n| = |B_{n0}| + |B_{n1}| + \cdots + |B_{nn}| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}. \hfill (2)$$

The set $A$ of all $k$-subsets of $[n]$ can be partitioned into the sets $B$ and $C$, where $B$ contains all the $k$-subsets which contain the element $n$, and $C$ contains all the $k$-subsets which do not contain $n$. For instance, for $n = 5$ and $k = 3$, we have

- $B = \{\{1, 2, 5\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}$,
- $C = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$.

The sum rule tells us that $\binom{n}{k} = |A| = |B| + |C|$. Deleting the element $n$ from every set in $B$ provides a bijection between $B$ and the $(k-1)$-subsets of $[n-1]$, hence $|B| = \binom{n-1}{k-1}$. The set $C$ is just the set of $k$-subsets of $[n-1]$, hence $|C| = \binom{n-1}{k}$, and by putting these things together we get the recursion

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \hfill (3)$$

Putting the binomial coefficients into a big table where the rows are indexed by $n$ and the columns are indexed by $k$, we obtain \textit{Pascal’s triangle}:

$$
\begin{array}{cccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 & \cdots \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 & \cdots \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & \cdots \\
\cdots
\end{array}
$$

Equation (1) says that the non-zero entries in each row are symmetric, equation (2) says that the sum of the entries in row $n$ is $2^n$ (we start counting rows with 0), and (3) tells us how to compute row $n$ from row $n-1$.

Now let $m$, $n$ and $r$ be non-negative integers, and let’s count the $r$-subsets of the set $[m+n] = \{1, 2, \ldots, m+n\}$. Let $A$ be the set of these subsets. By definition, $|A| = \binom{m+n}{r}$. Now let’s count in a different way. For $k = 0, 1, \ldots, r$, let $B_k$ be the set of $r$-subsets of $[m+n]$ which have $k$ elements in $[m]$. Clearly, the sets $B_k$ form a partition of $A$, hence $|A| = |B_0| + |B_1| + \cdots + |B_r|$ by the general sum rule. There are $\binom{m+n}{k}$ ways to
choose \( k \) elements from \([m]\), and \( \binom{n}{r-k} \) ways to choose \( r-k \) elements from \( \{m+1, m+2, \ldots, m+n\} \). This gives \( |B_k| = \binom{m}{k} \binom{n}{r-k} \), and we obtain the\textit{Vandermonde’s identity}:

\[
\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}.
\]

This is an illustration of an extremely important general principle called \textit{counting in two ways}: Whenever we count the elements of a certain set in two different ways, then the results must be the same.

**Lattice paths**

Very popular objects for counting are \textit{lattice paths}. Figure 3 shows a path from \((0, 0)\) to \((4, 4)\), where in each step we are moving either north or east.

![Figure 3. A lattice path from \((0, 0)\) to \((4, 4)\).](image)

How many lattice paths are there from \((0, 0)\) to \((m, n)\)? In total we have to make \( m+n \) steps, more precisely, \( m \) steps east and \( n \) steps north. The path in Figure 3 corresponds to the sequence “NEEENNEN” where “N” and “E” indicate north and east steps, respectively. A path is uniquely determined by specifying the positions of the north steps. For instance, we can reconstruct the path in Figure 3 if we know that steps 1, 5, 6 and 8 are north steps (the remaining steps 2, 3, 4 and 7) must be east steps. This gives a bijection between the lattice paths from \((0, 0)\) and the \( n \)-subsets of \([m+n]\), and the bijection rule tells us that the number of such lattice paths is \( \binom{m+n}{n} \).

Let’s assume \( m = n \), and let’s make it more interesting by adding the constraint that the path should never be above the diagonal \( x = y \). Figure 4 shows one path that satisfies this condition and one path that doesn’t.

What is the number of good paths from \((0, 0)\) to \((n, n)\)? For \( n = 1, 2, 3, 4 \), we can simply draw all the good paths and count them. The resulting numbers are 1, 2, 5, and 14. For \( n = 5 \) this is going to be tedious, and for \( n \geq 6 \) hopeless. Instead we can try to apply our counting techniques. An important observation is that instead of counting good paths we might as well count bad paths because we already know that there are \( \binom{2n}{n} \) paths in total, so the number of good paths is \( \binom{2n}{n} \) minus the number of bad paths. The trick for counting the bad paths is to define a bijection between the set of bad paths and a set that’s easier to count. To do this we take a bad path, identify its first step that moves above the diagonal, and complement each step after this (that is swap east and north after this first bad step). In Figure 5 this is illustrated for a bad path from \((0, 0)\) to \((4, 4)\). This defines a bijection between the set of bad paths from \((0, 0)\) to \((n, n)\) and the set of all paths from \((0, 0)\) to \((n-1, n+1)\), and by the bijection rule, we conclude that there are \( \binom{2n}{n+1} \) bad paths, and as a consequence, the number of good paths is

\[
\binom{2n}{n} - \binom{2n}{n+1} = \frac{2n}{n+1} - \frac{n}{n+1} \binom{2n}{n} = \frac{1}{n+1} \binom{2n}{n+1}.
\]

The numbers \( C_n = \frac{1}{n+1} \binom{2n}{n} \) are called \textit{Catalan numbers} and count an incredibly diverse range of objects.
COUNTING

(4, 4)

(0, 0)

Figure 4. A good path (left) and a bad path (right)

(4, 4)

(0, 0)

Figure 5. Turning a bad path from (0, 0) to (4, 4) into a path from (0, 0) to (3, 5). The green part of the bad path on the left is replaced by the red part of the path on the right.

The inclusion-exclusion principle

How many numbers of the set \{1, 2, \ldots, 10000\} are not divisible by any of the numbers 3, 5 and 7? Well, there are 10000 numbers in total, and of these

- 3333 are divisible by 3: \(1 \times 3 = 3, 2 \times 3 = 6, \ldots, 3333 \times 3 = 9999\),
- 2000 are divisible by 5: \(1 \times 5 = 5, 2 \times 5 = 10, \ldots, 2000 \times 5 = 10000\), and
- 1428 are divisible by 7: \(1 \times 7 = 7, 2 \times 7 = 14, \ldots, 1428 \times 7 = 9996\).

We might guess that there are \(10000 - 3333 - 2000 - 1428 = 3239\) numbers not divisible by 3, 5 or 7. The problem with this argument is that we have over-counted the bad guys because the numbers divisible by \(15 = 3 \times 5\), \(21 = 3 \times 7\) or \(35 = 5 \times 7\) have been subtracted more than once. Let’s try to correct this. There are

- 666 numbers divisible by 15: \(1 \times 15 = 15, 2 \times 15 = 30, \ldots, 666 \times 15 = 9990\),
- 476 numbers divisible by 21: \(1 \times 21 = 21, 2 \times 21 = 42, \ldots, 476 \times 21 = 9996\), and
- 285 numbers divisible by 35: \(1 \times 35 = 35, 2 \times 35 = 70, \ldots, 285 \times 35 = 9975\).

So our next guess is \(10000 - 3333 - 2000 - 1428 + 666 + 476 + 285 = 4720\). Fix any bad number \(k \in \{1, 2, \ldots, 10000\}\), that is, a number \(k\) which is divisible by at least one of the numbers 3, 5 or 7.

- If \(k\) is divisible by exactly one of the numbers 3, 5, 7, then it has been subtracted once (and not added again, because it’s not divisible by 15, 21 or 35).
If \( k \) is divisible by exactly two of the numbers 3, 5, 7, then it has been subtracted twice and added back once, because it’s divisible by exactly one of the numbers 15, 21, 35.

If \( k \) is divisible by all three of the numbers 3, 5, 7, then it has been subtracted three times and added back three times, because it’s divisible by all three of the numbers 15, 21, 35.

Our current guess does the right thing for the numbers divisible by one or two of the numbers 3, 5, 7, but it does not take into account the numbers divisible by all three, or equivalently, divisible by 3 \( \times \) 5 \( \times \) 7 = 105. Fortunately, this is easy to fix. There are 95 such numbers: 1 \( \times \) 105 = 105, 2 \( \times \) 105 = 210, \ldots, 95 \( \times \) 105 = 9975, and we conclude that the correct count is

\[
\]

This is an illustration of the inclusion-exclusion principle, which can be formulated in general as follows.

**Inclusion-exclusion principle:** If \( A = A_1 \cup A_2 \cup \cdots \cup A_n \), then

\[
|A| = |A_1| + |A_2| + \cdots + |A_n| - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n| + |A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| - \cdots
\]

In our example above, the set \( A \) of bad numbers can be written as \( A = A_1 \cup A_2 \cup A_3 \), where \( A_1 \) is the set of numbers divisible by 3, \( A_2 \) the set of numbers divisible by 5, and \( A_3 \) the set of numbers divisible by 7. Then the total number of bad numbers is

\[
|A| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| = 3333 + 2000 + 14428 - 666 - 476 - 285 + 95 = 5375.
\]

As another example, let’s determine the number of permutations of the set \( \{1, 2, \ldots, n\} \) which don’t have a fixed point, that is, we want to count the permutations \( (\pi_1, \pi_2, \ldots, \pi_n) \) of the numbers 1 to \( n \) with the property that \( \pi_i \neq i \) for every \( i \). For instance, for \( n = 1 \) there is no such permutation, for \( n = 2 \) there is exactly one:\n
\( (2, 1) \), for \( n = 3 \) there are two: \( (2, 3, 1) \) and \( (3, 1, 2) \), and for \( n = 4 \) there are nine:\n
\( (2, 1, 4, 3), (2, 3, 4, 1), (2, 4, 1, 3), (3, 1, 4, 2), (3, 4, 1, 2), (3, 4, 2, 1), (4, 1, 2, 3), (4, 3, 1, 2), (4, 3, 2, 1) \).

Let \( A \) be the set of permutations of \( [n] \) which have at least one fixed point. Then \( A = A_1 \cup A_2 \cup \cdots \cup A_n \), where \( A_i \) is the set of permutations which have \( i \) as a fixed point: \( A_i = \{(\pi_1, \ldots, \pi_n) : \pi_i = i\} \). For every \( i \), \( |A_i| = (n-1)! \), and for any distinct \( i \) and \( j \), \( A_i \cap A_j \) is the set of permutations fixing \( i \) and \( j \), hence \( |A_i \cap A_j| = (n-2)! \). More generally, for any subset \( I \subseteq [n] \), the intersection \( \bigcap_{i \in I} A_i \) is the set of permutations \( (\pi_1, \ldots, \pi_n) \) with \( \pi_i = i \) for all \( i \in I \), hence \( |\bigcap_{i \in I} A_i| = (n - |I|)! \), and the inclusion-exclusion principle gives

\[
|A| = n! \left( 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^{n-1} \frac{1}{n!} \right).
\]

As a consequence, the number of fixed point free permutations of \( [n] \) is

\[
a_n = n! - |A| = n! \left( \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^{n-1} \frac{1}{n!} \right).
\]

For instance,

\[
a_4 = 24 \left( \frac{1}{2} - \frac{1}{6} + \frac{1}{24} \right) = 12 - 4 + 1 = 9,
\]

\[
a_5 = 120 \left( \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} \right) = 60 - 20 + 5 - 1 = 44,
\]

\[
a_6 = 720 \left( \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \right) = 360 - 120 + 30 - 6 + 1 = 265.
\]
COUNTING Problems

(1) (a) How many zero/one sequences of length 20 are there which do not contain three zeros in a row?
(b) How many zero/one sequences of length 100 contain exactly 40 ones?

(2) (a) How many 3-subsets of the set \{1, 2, 3, 4, 5, 6, 7\} do not contain two consecutive numbers?
(b) For positive integers \(k \leq n\), how many \(k\)-subsets of \{1, 2, ..., n\} do not contain two consecutive numbers?

(3) (a) Consider a row of \(n\) seats. A child sits on each. Each child may move by at most one seat. In how many ways can they be rearranged?
(b) Consider a circular row of \(n\) seats. A child sits on each. Each child may move by at most one seat. In how many ways can they be rearranged?

(4) Prove that for positive integers \(n\) and \(r\),
\[
\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r}.
\]

(5) \(2n\) points are chosen on a circle. In how many ways can you join pairs of points by nonintersecting chords?

(6) Let \(a_n\) be the numbers of permutations \((\pi_1, \pi_2, \ldots, \pi_{2n})\) of the set \{1, 2, \ldots, 2n\} such that \(\pi_i + \pi_{i+1} \neq 2n+1\) for all \(i \in \{1, \ldots, 2n-1\}\). In this problem you will derive a formula for \(a_n\).
(a) For \(k \in \{0, \ldots, n\}\), prove that there are \(\binom{2n-k}{k}\) subsets \(X = \{x_1, x_2, \ldots, x_k\} \subseteq \{1, \ldots, 2n-1\}\) such that \(x_i \geq x_{i-1} + 2\) for all \(i \in \{2, \ldots, k\}\).
(b) Let \(X = \{x_1, x_2, \ldots, x_k\} \subseteq \{1, \ldots, 2n-1\}\) with \(x_i \geq x_{i-1} + 2\) for all \(i \in \{2, \ldots, k\}\). How many permutations \((\pi_1, \ldots, \pi_{2n})\) have the property that \(\pi_i + \pi_{i+1} = 2n + 1\) for all \(i \in X\)?
(c) Use the inclusion-exclusion principle to write down a formula for \(a_n\). Try to simplify the resulting expression as much as possible.