CONSTRaining KALMAN Filter and SMOOTHing ESTimates to SATISfy TIME-VARYING RESTricTIONS

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Constraining Kalman Filter and Smoothing Estimates to Satisfy Time-Varying Restrictions.

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Sometimes a priori restrictions exist on the state vector of a linear dynamic model. It is shown that when these restrictions are linear, the Kalman filter and smoothing estimates can be made to satisfy them by augmenting the observation equation. When the restrictions are non-linear, an iterative procedure is available.

Keywords: Kalman filter; Kalman smoothing; linear constraints; non-linear constraints; artificial observations.
1. **Introduction**

Application of the Kalman filtering and smoothing techniques has become standard for the estimation of linear state space models.

It may sometimes happen that additional information is available in the form of time-varying constraints on the state vector. Incorporation of this information will, of course, increase the efficiency of estimation.

In this note, it is shown that such constraints can always be incorporated in the estimation by augmenting the observation equation by artificial observations. A suggestion is made about how the methodology could be extended to satisfy non-linear constraints.

2. **Linear Constraints on Kalman Filter Estimates**

Consider the linear dynamic system

transition equation: \( \alpha_t = A_t \alpha_{t-1} + \eta_t \)  \( (2.1) \)

observation equation: \( Y_t = Z_t \alpha_t + \xi_t \)  \( (2.2) \)

where \( E(\xi_t' \xi_t) = H_t \) and \( E(\eta_t' \eta_t) = Q_t \). Here \( \alpha_t \) is an \( m \times 1 \)
unobserved state vector and \( y_t \) and \( n \times 1 \) vector of observations. It is assumed that \( A_t, Q_t, Z_t \) and \( H_t \) are known matrices of dimensions \( m \times m, m \times m, n \times m \) and \( n \times n \), respectively. Furthermore, it is assumed that \( E(\xi_s \eta_t') = 0 \) for all \( t \) and \( s \).

A recursive solution to the estimation of the state vector \( \alpha_t \) was originally given by Kalman (1960) and has since been used in numerous contexts. In what follows, Harvey's (1981) notation is used.

Let us suppose that it is known that the state vector \( \alpha_t \) obeys the time-varying linear constraint

\[
R_t \alpha_t = r_t,
\]

where \( R_t \) and \( r_t \) are known and of dimensions \( j_t \times m \) and \( j_t \times 1 \), respectively. Notice that not only the constraints, but also the number of constraints, may vary with time.

**Theorem 1:** If the observation equation (2.2) is augmented by the \( j_t \) artificial 'observations'

\[
r_t = R_t \alpha_t, \tag{2.3}
\]

then the Kalman filter estimates \( \hat{\alpha}_t \) also satisfy the linear constraints. That is,
Proof

In Harvey's (1981) notation, $\hat{\alpha}_t$ is given by

$$\hat{\alpha}_t = \hat{\alpha}_{t|t-1} + G_t (y_t - Z_t \hat{\alpha}_{t|t-1}),$$  \hspace{1cm} (2.5)

where

$$\hat{\alpha}_{t|t-1} = A_t \hat{\alpha}_{t-1},$$

$$G_t = P_{t|t-1} Z_t' F_t^{-1},$$

and

$$F_t = Z_t P_{t|t-1} Z_t' + H_t.$$  \hspace{1cm} (2.6)

The matrix $P_{t|t-1}$ is $m \times m$ symmetric positive semi-definite, but its definition need not concern us here.

Let us now augment the observation equation (2.2) by the artificial 'observation' (2.3). That is, we define a new observation equation

$$y_t^* = Z_t^* \alpha_t + \xi_t^*,$$  \hspace{1cm} (2.6)

where
\[ y_t^* = \begin{bmatrix} y_t \\ r_t \end{bmatrix}, \quad z_t^* = \begin{bmatrix} z_t \\ R_t \end{bmatrix} \quad \text{and} \quad \xi_t^* = \begin{bmatrix} \xi_t \\ 0 \end{bmatrix}. \]

It should be noted that \( E(\xi_t^*\xi_t'^*) = \begin{bmatrix} H_t & 0 \\ 0 & 0 \end{bmatrix}. \)

In order to establish the theorem, it will be necessary to consider \( R_t \) \( G_t \) for the augmented system. Now

\[ R_tG_t = R_tP_t|t-1Z'_tF_t^{-1} = R_tP_t|t-1[Z'_t,R_t] \begin{bmatrix} F_{11}' & F_{12}' \\ F_{21}' & F_{22}' \end{bmatrix} \]

\[ = [(R_tP_t|t-1Z'_t)F_{11}' + (R_tP_t|t-1R'_t)F_{21}'] + (R_tP_t|t-1Z'_t)F_{12}' + (R_tP_t|t-1R'_t)F_{22}' \quad (2.7) \]

where \( F_t^{-1} \) has been partitioned in the obvious way, and the \((i, j)\) block denoted by \( F_{ij} \). With the augmented system,
Using standard theorems on partitioned inverses (see, for example, Theil (1971, 17-18))

\[ F_{21} = - (R_t P_t | t-1 R_t')^{-1} R_t P_t | t-1 Z_t' F_{11} \]

and so

\[ (R_t P_t | t-1 Z_t') F_{11} + (R_t P_t | t-1 R_t) F_{21} = 0. \] (2.8)

Also,

\[ F_{12} = - (Z_t P_t | t-1 Z_t + H_t)^{-1} Z_t P_t | t-1 R_t' F_{22} \]

and

\[ F_{22} = \left[ R_t P_t | t-1 R_t - R_t P_t | t-1 Z_t (Z_t P_t | t-1 Z_t + H_t)^{-1} \right]^{-1}. \]
These two latter equations yield

\[(R_t P_t | t-1 Z'_t) F_{12} + (R_t P_t | t-1 R'_t) F_{22} = I_j_t. \]  (2.9)

Thus, using (2.8) and (2.9) with (2.7),

\[R_t G_t = [0, I_j_t]. \]  (2.10)

By equation (2.5), for the augmented system

\[R_t \hat{\alpha}_t = R_t \hat{\alpha}_{t|t-1} + R_t G_t(Y_t^* - Z_t \hat{\alpha}_t | t-1)\]

\[= R_t \hat{\alpha}_{t|t-1} + [0, I_j_t] \begin{bmatrix} Y_t - Z_t \hat{\alpha}_t | t-1 \\ r_t - R_t \hat{\alpha}_t | t-1 \end{bmatrix} \]

\[= R_t \hat{\alpha}_{t|t-1} + (r_t - R_t \hat{\alpha}_t | t-1) = r_t. \]

Therefore,

\[R_t \hat{\alpha}_t = r_t, \]

and the theorem is proved.
3. **Kalman Smoothing**

The filter estimate $\hat{a}_t$ only utilizes information up to period $t$. The only estimate which uses all the sample information for $t=1,2,...,T$ is $\hat{a}_T$. The smoothing algorithm works backwards from $t=T$, adjusting $\hat{a}_t$ to obtain optimal estimates in the light of the whole sample. We will donate the smoothed estimator by $\hat{a}_t|T$. The smoothing algorithm (see Harvey (1981, 115)) is given by

\begin{align}
\hat{a}_t|T &= \hat{a}_t + P_t^*(\hat{a}_{t+1}|T - A_{t+1}\hat{a}_t), \quad (3.1) \\
P_t|T &= P_t + P_t^*(P_{t+1}|T - P_{t+1}|t) P_t^*, \quad (3.2) \\
P_t^* &= P_t A_{t+1} P_{t+1|t}^{-1} \quad (3.3) \\
\text{and} \\
P_t &= P_t|t-1 - G_t Z_t P_t|t-1. \quad (3.4)
\end{align}

**Theorem 2:**

When the augmented observation equation is used, the
smoothed estimates also satisfy the linear constraints. That is,

\[ R_t \hat{\alpha}_t | T = r_t. \]  

(3.5)

**Proof:**

As \( R_t \hat{\alpha}_t = r_t \) (Theorem 1), from (3.1) a sufficient condition for (3.5) is that \( R_t P_t^* = 0 \). From (3.3) a sufficient condition for \( R_t P_t^* = 0 \) is that \( R_t P_t = 0 \).

Now by (3.4), with the augmented system,

\[ R_t^P_t = R_t P_t | t-1 - (R_t G_t) Z_t^* P_t | t-1 \]

\[ = R_t P_t | t-1 - \begin{bmatrix} 0, I \end{bmatrix} \begin{bmatrix} Z_t \\ R_t \end{bmatrix} P_t | t-1 \]

\[ = 0. \]

Thus,

\[ R_t \hat{\alpha}_t | T = r_t, \]

and the result is proved.
4. An Example

In an attempt to estimate the populations of the \( k = 8 \) states and territories in Australia between censuses, Doran (1990) used a model of the form

\[
X_{it} = X_{i,t-1} + b_{it} + c_{it} + \eta^{(1)}_t,
\]

\[
b_{it} = \mu_i + b_{i,t-1} + \eta^{(2)}_t,
\]

where \( X_{it} \) is the population of the \( i \)th state or territory \( (i = 1, 2, \ldots, k) \), \( c_{it} \) the observed natural increase during period \( t \) and \( b_{it} \) a growth factor, due largely to net migration.

Defining

\[
\alpha_t = [X_{1t}, X_{2t}, \ldots, X_{kt}; b_{1t}, b_{2t}, \ldots, b_{kt}]',
\]

\[
\beta_t = [c_{1t}, c_{2t}, \ldots, c_{kt}; 0, 0, \ldots, 0]',
\]

\[
\mu = [0, 0, \ldots, 0; \mu_1, \mu_2, \ldots, \mu_k]',
\]

\[
\eta_t = [\eta^{(1)}_1, \eta^{(1)}_2, \ldots, \eta^{(1)}_k; \eta^{(2)}_1, \eta^{(2)}_2, \ldots, \eta^{(2)}_k]'.
\]

the model can be written in the state transition form
\[ \alpha_t = A \alpha_{t-1} + \mu + c_t + \eta_t, \]

where

\[ A = \begin{bmatrix} I_k & I_k \\ 0 & I_k \end{bmatrix}. \]

The variables \( X_{it} \) are observed in census years, and sum to the observed national total in non-census years. Thus,

\[ R_t \alpha_t = r_t, \]

where, in census years,

\[ R_t = [I_k, 0], \quad r_t = [X_{1t}, X_{2t}, \ldots, X_{kt}]'. \]

and in non-census years

\[ R_t = [1, 1, \ldots, 1; 0, 0, \ldots, 0], \quad r_t = \sum_{i=1}^{k} X_{it}. \]

In all years, \( R_t \) is known and \( r_t \) observed.

By including the 'observation equation'

\[ r_t = R_t \alpha_t, \]

with \( R_t \) and \( r_t \) as above, Doran ensured that the filter predictions \( \hat{X}_{it} \) and the smoothed predictions \( \hat{X}_{it|T} \) satisfied the constraints.
5. **Non-Linear Constraints**

The foregoing suggests an iterative procedure for incorporating non-linear constraints.

Assume that at time \( t \) the state vector \( \alpha_t \) satisfies \( j_t \) non-linear constraints

\[
    f_{it}(\alpha_t) = 0, \quad i=1,2,\ldots,j_t.
\]

That is,

\[
f_t(\alpha_t) = 0, \quad \text{(5.1)}
\]

where \( f_t(.) \) is a vector function of dimension \( j_t \).

For example, in the Doran (1990) application, it may have been reasonable to express the population figures in logarithms, so that the \( b_{it} \) represented growth rates. Under this transformation, the non-census years constraint would have been

\[
    \sum_{i=1}^{k} \exp(X_{it}) = r_t,
\]

where \( r_t \) is the (observed) national population in year \( t \). That is,
\[ f_t(\alpha_t) = \sum_{i=1}^{k} \exp(v_i' \alpha_t) - r_t, \]

where \( v_i \) is an \( m \)-vector with one in the \( i \)th position and zeroes elsewhere.

Suppose now that the model (2.1) and (2.2) has been estimated by the Kalman procedure, and has yielded an unconstrained estimate \( \hat{\alpha}_t^{(0)} \). Assuming that \( \hat{\alpha}_t^{(0)} \) is 'close' to \( \alpha_t \), we have by Taylor's expansion

\[ 0 = f_t(\alpha_t) \approx f_t(\hat{\alpha}_t^{(0)}) + Df_t(\hat{\alpha}_t^{(0)})(\alpha_t - \hat{\alpha}_t^{(0)}), \]

where \( Df_t(.) \) is the Jacobian matrix of \( f_t(.) \).

Thus,

\[ Df_t(\hat{\alpha}_t^{(0)})\alpha_t \approx Df_t(\hat{\alpha}_t^{(0)})\hat{\alpha}_t^{(0)} - f_t(\hat{\alpha}_t^{(0)}). \]

It follows that if we define

\[ R_t = Df_t(\hat{\alpha}_t^{(0)}), \tag{5.2} \]

\[ r_t = Df_t(\hat{\alpha}_t^{(0)})\hat{\alpha}_t^{(0)} - f_t(\hat{\alpha}_t^{(0)}), \tag{5.3} \]

and augment the observation equation as discussed above, then estimates will be obtained which approximately satisfy
the constraints $f_t(\hat{\alpha}_t) = 0$.

Using $\hat{\alpha}_t$ now as the starting point, the process is repeated until convergence. Of course, as with all iterative procedures of this type, the first estimate $\hat{\alpha}_t^{(0)}$ is crucial if convergence is to occur.
References

Doran, H.E. (1990), "Using the Kalman Filter to Estimate Sub-populations," Working papers in Econometrics and Applied Statistics 44 (University of New England, Armidale)


A Note on A Bayesian Estimator in an Autocorrelated Error Model. William Griffiths and Dan Dao, No. 3 - April 1979.


Bayesian Econometrics and How to Get Rid of Those Wrong Signs. William E. Griffiths, No. 31 - November 1987.


