5 Linear Equations

5.1 Solution of a linear Equation with one Variable

A linear equation contains no indices greater than 1 of the variable \( x \) (say), e.g. \( 3x - 5 = 7 \) is a linear equation but \( 3x^2 - 5 = 7 \) is a non-linear equation.

A linear equation can be represented pictorially as a straight line – see Topic 6.

The two sides of an equation are like the two sides of a set of scales in equilibrium. Whatever you do to one side you must do to the other side to preserve the equilibrium.

\[
\text{L.H.S.} = \text{R.H.S.}
\]

Examples:

(a) Solve \( 3x - 5 = 7 \)

Solution: 
\[
3x - 5 + 5 = 7 + 5 \quad \text{adding 5 to each side}
\]
\[
3x = 12 \quad \text{simplifying}
\]
\[
\frac{3x}{3} = \frac{12}{3} \quad \text{dividing both sides by 3}
\]
\[
x = 4 \quad \text{simplifying}
\]

Check by substituting \( x = 4 \) in the original equation

\[
\text{i.e.} \quad \text{L.H.S.} = 3 \times 4 - 5 = 12 - 5
\]
\[
= 7
\]
\[
= \text{R.H.S.}
\]
\[
\therefore \quad x = 4 \quad \text{is the correct solution.}
\]
(b) Solve $4(x - 1) + 2 = 2x - 5$

Solution:

$$4x - 4 + 2 = 2x - 5$$
$$4x - 2 = 2x - 5$$
$$4x - 2 + 2 = 2x - 5 + 2$$
$$4x = 2x - 3$$
$$4x - 2x = 2x - 3 - 2x$$
$$2x = -3$$
$$\frac{2x}{2} = \frac{-3}{2}$$
$$x = -\frac{1}{2}$$

Check:

L.H.S. = $4 \left( -\frac{1}{2} - 1 \right) + 2$ when $x = -\frac{1}{2}$

$$= 4 \left( -\frac{3}{2} \right) + 2$$
$$= -10 + 2$$
$$= -8$$

R.H.S. = $2 \left( -\frac{1}{2} \right) - 5$ when $x = -\frac{1}{2}$

$$= -3 - 5$$
$$= -8$$

= L.H.S.

$\therefore x = -\frac{1}{2}$ is the correct solution.

Note: While checking is seldom required in a problem, it is a useful exercise especially if you are asked to use the answer in some further problem. So, make a habit of checking.
Exercises 5.1:
Solve
(i) $3x - 1 = 5$
(ii) $4x + 2 = 9$
(iii) $6 - 2x = 1$
(iv) $5x + 1 = -9$
(v) $3x + 1 = x - 5$
(vi) $6x - 1 = 3 - 2x$
(vii) $2 - 3(x - 2) = x + 4$
(viii) $5(x - 1) + 1 = 2(x - 2)$.

In general, a linear equation in one variable has just one solution, i.e., a unique solution.

[Examples of linear equations in one variable which do not have a unique solution are

$$3\left(x + \frac{4}{3}\right) = 3x + 4 \quad \text{and} \quad 3x + 4 = 3(x + 4).$$

In the first equation, both sides of the equation are identical and so any value of $x$ will satisfy the equation and hence there will be an “infinite number” of solutions. In the second equation, $3x + 4 = 3x + 12$, which is a contradiction, so no solution is possible.]

A linear equation in more than one variable will not have a unique solution, e.g. $3x + 4y = 2$ (see Section 2 below).
5.2 Linear Equations in two Variables

Examples:

(a) \( 3x + 4y = 0 \) i.e. \( y = -\frac{3}{4}x \) has an “infinite number” of solutions which may be tabulated thus:

| \( x \) | \( \cdots \) | \(-1\) | 0 | 1 | \( \frac{4}{3}\) | 2 | \( \cdots \) |
| \( y \) | \( \cdots \) | \( \frac{3}{4} \) | 0 | \( -\frac{3}{4} \) | -1 | \(-\frac{1}{2}\) | \( \cdots \) |

For every value of \( x \), there is a value of \( y \), e.g., \( x = \frac{4}{3}, \ y = -1 \).

(b) \( 3x + 4y = 2 \) i.e. \( y = \frac{1}{2} - \frac{3x}{4} \) has an infinite number of solutions, e.g.

| \( x \) | \( \cdots \) | \(-1\) | 0 | 1 | \( \frac{4}{3}\) | 2 | \( \cdots \) |
| \( y \) | \( \cdots \) | \( 1\frac{1}{4} \) | \( \frac{1}{2} \) | \(-\frac{1}{4}\) | \(-\frac{1}{2}\) | -1 | \( \cdots \) |

Exercise 5.2: Solve \( 2x - y = 3 \)

5.3 Pair of Linear Equations in two Variables

Suppose we are given the equations \( 3x + 4y = 2 \) and \( x + y = 3 \). Each equation separately has an infinite number of solutions. Taken together, however, these two equations have a unique solution.

The equations are then said to be solved simultaneously i.e., “at the same time”.

There are two equivalent methods of solution to this problem.

Example:

\[
\begin{align*}
\text{Solve} & \quad 3x + 4y = 2 \quad \text{...............} \quad (i) \\
\text{simultaneous linear equations} & \quad x + y = 3 \quad \text{...............} \quad (ii) \\
\text{linear equations} & \quad \{ \}
\end{align*}
\]

First solution:

\[
\begin{align*}
(ii) \times (4) & : \quad 4x + 4y = 12 \quad \text{...............} \quad (iii) \\
3x + 4y & = 2 \quad \text{...............} \quad (i) \\
(iii) - (i) & : \quad x = 10
\end{align*}
\]
Substitute in (ii):

\[
10 + y = 3
\]
\[
10 + y - 10 = 3 - 10
\]
\[
y = -7
\]

\[
\begin{align*}
x &= 10 \\
y &= -7
\end{align*}
\]

∴ the unique solution is

Always substitute these values of \( x \) and \( y \) in (i), (ii) to check the correctness of the solution.

Second solution: Write (i), (ii) as

\[
y = \frac{-3}{4}x + \frac{1}{2} \quad \ldots \ldots \ldots \ldots \ldots \ldots (i)' \\
y = -x + 3 \quad \ldots \ldots \ldots \ldots \ldots \ldots (ii)'
\]

\( (i)' \), \( (ii)' \) give \(-\frac{3}{4}x + \frac{1}{2} = -x + 3 \) (= \( y \)) which we solve as in Topic 5, Section 1.

\[
\begin{align*}
-\frac{3}{4}x + \frac{1}{2} - \frac{1}{2} &= -x + 3 - \frac{1}{2} \\
-\frac{3}{4}x &= -x + 2\frac{1}{2} \\
-\frac{3}{4}x + x &= -x + 2\frac{1}{2} + x \\
\frac{1}{4}x &= 2\frac{1}{2} \\
\frac{1}{4}x \times 4 &= 2\frac{1}{2} \times 4 \\
x &= 10
\end{align*}
\]

Substitute for \( x \) in \( (i)' \): \( y = -\frac{3}{4} \times 10 + \frac{1}{2} = -7\frac{1}{2} + \frac{1}{2} = -7 \)

Substitute for \( x \) in \( (ii)' \): \( y = -10 + 3 = -7 \)

∴ the unique solution is

\[
\begin{align*}
x &= 10 \\
y &= -7
\end{align*}
\]

[The checking of the solution is inherent in the process.]
Exercises 5.3:

(i) Solve

(a) \(2x + 3y = 10, \ 3x - 2y = -11\)
(b) \(5x - 2y = 1, \ 2x = 4 - y\)

(ii) Why do the pairs of equations

(a) \(x - y = 5, \ 3x - 3y = 4\)
(b) \(x - y = 5, \ 3x - 3y = 15\)

not have a unique solution?

5.4 Geometrical Interpretation

The *unique* solution of 2 simultaneous *linear* equations gives the *unique* point of intersection of 2 straight *lines*.

See the next Topic, Graphs of Straight Lines.
5.5 Answers to Exercises

5.1:

(i) 2  
(ii) $\frac{3}{4}$  
(iii) $2\frac{1}{2}$  
(iv) $-2$  
(v) $-3$  
(vi) $\frac{1}{2}$  
(vii) 1  
(viii) 0

5.2:

\[
\begin{align*}
  x & = \ldots, -1, 0, 1, 1\frac{1}{2}, 2, \ldots \\
  y & = \ldots, -5, -3, -1, 0, 1, \ldots
\end{align*}
\]

are some solutions.

5.3:

(i) (a) $x = -1$, $y = 4$  
(b) $x = 1$, $y = 2$  

(ii) (a) the two equations are inconsistent. (if $x - y = 5$ is true, then $3x - 3y = 3(x - y) = 3 \times 5 = 15$, not 4)  
(b) the second equation is exactly 3 times the first equation, i.e., there is only one (independent) equation, i.e., the equations are not independent.